

# EPISTEMIC EXTENSION OF GÖDEL LOGIC

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**ABSTRACT.** In this paper, some epistemic extensions of Gödel logic are introduced. We establish two proof systems and show that these logics are sound with respect to an appropriate Kripke semantics. Furthermore, we demonstrate weak completeness theorems, that is if a formula  $\varphi$  is valid then its double negation  $\neg\neg\varphi$  is provable. A fuzzy version of muddy children puzzle is given and using this, it is shown that the positive and negative introspections are not valid. We enrich the language of epistemic Gödel logic with two connectives for group and common Knowledge and give corresponding semantics for them. We leave the problem of soundness and completeness in this general setting open. Finally, an action model approach is introduced to establish a dynamic extension of epistemic Gödel logic.

## 1. INTRODUCTION

Many modal extensions of fuzzy logics have been introduced in the literature. In [3, 4, 5, 6], some modal extensions of Gödel fuzzy logic are presented. Some modal extensions of Lukasiewicz logic are studied in [16, 17], where a classical accessibility relation semantics is used. In [22], some modal extensions of product fuzzy logic by using both relational and algebraic semantics are studied. The relational semantics of these extensions is based on Kripke structures with classical accessibility relations. In [11], Hajek proposes a fuzzy variant of each recursively axiomatized logic extending  $S_5$ . [23] studies some modal logics over MTL, where the semantics is based on Kripke structures with truth values in  $[0,1]$  and classical accessibility relations.

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In this paper, some epistemic extensions of propositional Gödel logic are introduced with both fuzzy propositions and fuzzy accessibility relations. The main new fuzzification made in this paper, is fuzzifying the precondition function of an epistemic action in the introduced dynamic epistemic

In Section 2, we first propose a language for an epistemic extension of Gödel fuzzy logic and give a Kripke semantics while both propositions at the possible worlds and accessibility relations are fuzzy taking values in  $[0,1]$ . Accessibility relations are not necessarily symmetric, since the amount of indistinguishing may differ in different worlds. Also, a fuzzy version of popular muddy children puzzle is proposed to show that some formulas including *positive* and *negative introspection* are not valid. In the sequel of the section, we propose two axiom systems which are sound with respect to the corresponding semantics. Also, we obtain weak completeness theorems, in the sense that if a formula  $\varphi$  is valid then  $\neg\neg\varphi$  is provable. Section 3 is devoted to the soundness and weak completeness theorems.

In Section 4, we expand the language of epistemic Gödel logic with two connectives for group and common knowledge. The corresponding semantics is presented, but the problem of soundness and completeness are left open in this setting.

In Section 5, we introduce an action model approach to give a dynamic extension of epistemic Gödel logic. The language of an action model includes formulas with  $G(\varphi) = g$  and  $G(\varphi) > g$  notations. Intuitively, these formulas inform the agents about the truth values of pre-conditions. Furthermore, we show the validity of some formulas which are similar to some axioms of the proof system AMC [7]. Finally, by considering the agents with hearing impairment, we give an example that our fuzzy muddy children puzzle is updated after some announcements.

We assume that the reader is familiar with basic facts on the classical modal logic, see e.g. [1]. The main reference for the Gödel logic is [12]. In particular, we have the following theorems in Gödel logic which we will use in our proofs;

$$\begin{aligned}
&[(G2)] \quad (\varphi \wedge \psi) \rightarrow \varphi \\
&[(G4)] \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\varphi \wedge \psi) \rightarrow \chi) \\
&[(GT1)] \quad \psi \rightarrow (\varphi \rightarrow \psi) \\
&[(GT2)] \quad (\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi) \\
&[(GT3)] \quad \neg(\varphi \wedge \psi) \rightarrow (\varphi \rightarrow \neg\psi) \\
&[(GT4)] \quad (\neg\neg\varphi \wedge \neg\neg\psi) \leftrightarrow \neg\neg(\varphi \wedge \psi) \\
&[(GT5)] \quad \neg\neg(\varphi \rightarrow \psi) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)
\end{aligned}$$

$$\begin{aligned}
[(GT6)] \quad & \varphi \rightarrow \neg\neg\varphi \\
[(GT7)] \quad & \neg\varphi \leftrightarrow \neg\neg\neg\varphi \\
[(GT8)] \quad & \neg(\varphi \rightarrow \psi) \rightarrow (\neg\varphi \rightarrow \psi)
\end{aligned}$$

Throughout this paper, it is assumed that  $\mathcal{P}$  is a set of atomic propositions and  $\mathcal{A}$  is a set of agents, unless otherwise stated.

## 2. EPISTEMIC GÖDEL LOGIC

### 2.1. Semantics of epistemic Gödel logic.

**Definition 2.1.** The language of epistemic Gödel logic (EGL) is generated by the following BNF:

$$\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid B_a \varphi$$

where,  $p \in \mathcal{P}$  and  $a \in \mathcal{A}$ .

Note that the language of EGL is an expansion of the language of Gödel logic. It is enriched by all epistemic connectives  $B_a$ , where  $a \in \mathcal{A}$ . Further connectives  $\neg$ ,  $\vee$  and  $\leftrightarrow$  are defined as similar as GL [12].

**Definition 2.2. (EGL Model)** An EGL-model is a structure  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$ , where

- $S$  is a set of states,
- $r_{a|_{a \in \mathcal{A}}} : S \times S \rightarrow [0, 1]$  is a which assigns a value in  $[0, 1]$  to each  $(s, s') \in S \times S$ . We call it the *indistinguishing* function.
- $\pi : S \times \mathcal{P} \rightarrow [0, 1]$  is a *valuation* function which assigns a truth value to each atomic proposition  $p \in \mathcal{P}$ , in every state  $s \in S$ .

The valuation function  $\pi$ , can be extended to all formulas naturally, denoted by  $V$ . The model  $M$  is called *reflexive* if for all  $a \in \mathcal{A}$  and all  $s \in S$ ,  $r_a(s, s) = 1$ .

Let  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  be an EGL-model. For each state  $s \in S$  and each formula  $\varphi$ , we use the notation  $V_s(\varphi)$  for  $V(s, \varphi)$ , which is defined as follows:

- $V_s(p) = \pi(s, p)$ ;  $p \in \mathcal{P}$
- $V_s(\varphi \wedge \psi) = \min\{V_s(\varphi), V_s(\psi)\}$
- $V_s(\varphi \rightarrow \psi) = \begin{cases} 1 & V_s(\varphi) \leq V_s(\psi) \\ V_s(\psi) & V_s(\varphi) > V_s(\psi) \end{cases}$
- $V_s(\neg\varphi) = \begin{cases} 0 & V_s(\varphi) > 0 \\ 1 & V_s(\varphi) = 0 \end{cases}$
- $V_s(\varphi \vee \psi) = \max\{V_s(\varphi), V_s(\psi)\}$
- $V_s(B_a(\varphi)) = \min_{s' \in S} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\}$

Note that  $\neg\varphi$  and so  $\neg\neg\varphi$  take the crisp values.

**Definition 2.3.** Let  $\varphi$  be an EGL-formula and  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  be an EGL-model.

- (1) If  $s \in S$ , we say  $\varphi$  is valid in the *pointed model*  $(M, s)$ ; notation  $(M, s) \models \varphi$ , if  $V_s(\varphi) = 1$ .
- (2)  $\varphi$  is  $M$ -valid; notation  $M \models \varphi$ , if for each state  $s' \in S$ ,  $(M, s') \models \varphi$ .
- (3) If  $\mathcal{M}$  is a class of models, we say  $\varphi$  is  $\mathcal{M}$ -valid; notation  $\mathcal{M} \models \varphi$ , if for all  $M' \in \mathcal{M}$ ,  $M' \models \varphi$ .
- (4)  $\varphi$  is EGL-valid; notation  $\models \varphi$ , if for each EGL-model  $M^*$ ,  $M^* \models \varphi$ .

**Example 2.4. (A fuzzy muddy children)** Let  $\mathcal{A} = \{a_1, \dots, a_k\}$  be a set of children (agents) with muddy faces. The agents may be visually impaired. We consider  $n$  intervals with equal length except the first zero-length interval. Suppose that  $(\beta_i, \beta_{i+1}]$ ;  $0 \leq i \leq n-1$ ; be the desired intervals, which  $\beta_i$ s satisfy the following conditions:

$$\beta_0 = \beta_1 = 0, \quad \beta_{i+1} - \beta_i = \frac{1}{n-1} \quad (1 \leq i \leq n-1)$$

For each  $0 \leq i \leq n-1$ , we name the interval  $(\beta_i, \beta_{i+1}]$  by  $\beta_{i+1}$ , then let  $R = \{\beta_i \mid 1 \leq i \leq n\}$ . We consider  $S = \{(t_j)_{1 \leq j \leq k} \mid t_j \in R\}$  as the set of possible worlds. Also, corresponding to each agent  $a_j$ ,  $1 \leq j \leq k$ , we consider an atomic proposition  $m_j$ , which intuitively means "*the face of the agent  $a_j$  is muddy*". Let  $\mathcal{P} = \{m_j \mid 1 \leq j \leq k\}$  be the set of atomic propositions. Then, the valuation function  $\pi : S \times \mathcal{P} \rightarrow [0, 1]$  is defined as follows:

$$\pi((t_j)_{1 \leq j \leq k}, m_i) = t_i \quad (1 \leq i \leq k)$$

Suppose that  $\alpha \in (0, 1)$  and for each  $a \in \mathcal{A}$ ,  $B_a \in [0, 1]$  is the amount of *visual impairment* of the agent  $a$ . Also, let  $s_1 = (t_j^1)_{1 \leq j \leq k}$  and  $s_2 = (t_j^2)_{1 \leq j \leq k}$  be two states in  $S$ . Then, the indistinguishing functions are defined as follows:

$$r_{a_i a_j}^{s_1 s_2} = \begin{cases} B_{a_i} (1 - \alpha |t_j^1 - t_j^2|) & t_j^1 \neq t_j^2 \\ 1 & t_j^1 = t_j^2 \end{cases} \quad \text{where, } 1 \leq i, j \leq k, j \neq i$$

$$r_{a_i}(s_1, s_2) = \min \left\{ r_{a_i a_j}^{s_1 s_2} \mid 1 \leq j \leq k, j \neq i \right\} \quad \text{where, } 1 \leq i \leq k$$

Note that the indistinguishing functions defined above are compatible with our intuition, because the more the difference between the amounts of mud on the faces of the other agents, the easier it is to distinguish the states exactly.

**Proposition 2.5.** *The following schemes are not EGL-valid.*

- (1)  $B_a \varphi \rightarrow B_a B_a \varphi$
- (2)  $\neg B_a \varphi \rightarrow B_a (\neg B_a \varphi)$
- (3)  $B_a (\varphi \vee \psi) \rightarrow B_a \varphi \vee B_a \psi$
- (4)  $\neg B_a \varphi \rightarrow B_a \neg \varphi$
- (5)  $B_a \varphi \vee B_a \neg \varphi$

$$(6) \quad \varphi \rightarrow B_a \neg B_a \neg \varphi$$

*Proof.* We construct an EGL-model  $M'$  as the Example 2.4. Let  $n = 3$ ,  $\mathcal{A} = \{a, b\}$ ,  $\mathcal{B}_a = 0.4$ ,  $\mathcal{B}_b = 0.9$  and  $\alpha = 0.2$ . Then,  $S = \{(i, j) \mid i, j \in \{0, 0.5, 1\}\}$ , and for arbitrary states  $s_1 = (x, y)$  and  $s_2 = (x', y')$  in  $S$ , the indistinguishing functions are defined as follows:

$$r_a(s_1, s_2) = \begin{cases} 1 & |y - y'| = 0 \\ 0.36 & |y - y'| = 0.5 \\ 0.32 & |y - y'| = 1 \end{cases} \quad r_b(s_1, s_2) = \begin{cases} 1 & |x - x'| = 0 \\ 0.81 & |x - x'| = 0.5 \\ 0.72 & |x - x'| = 1 \end{cases}$$

It can be shown that some instances of the desired schemes fail to be valid in the model  $M'$ . For each part we find an state  $s \in S$  and a formula  $\varphi$  such that  $(M', s) \not\models \varphi$ .

(1): We show that the formula  $B_b m_a \rightarrow B_b B_b m_a$  is not valid in  $(M', (1, 1))$ .

The following statements hold:

$$\begin{aligned} V_{(0,0)}(B_b m_a) &= V_{(0,0.5)}(B_b m_a) = V_{(0,1)}(B_b m_a) = 0, \\ V_{(0.5,0)}(B_b m_a) &= V_{(0.5,0.5)}(B_b m_a) = V_{(0.5,1)}(B_b m_a) = 0.19, \\ V_{(1,0)}(B_b m_a) &= V_{(1,0.5)}(B_b m_a) = V_{(1,1)}(B_b m_a) = 0.28. \end{aligned}$$

Therefore,  $V_{(1,1)}(B_b B_b m_a) = 0.19$ , and because  $V_{(1,1)}(B_b m_a) > V_{(1,1)}(B_b B_b m_a)$ , it can be concluded that  $V_{(1,1)}(B_b m_a \rightarrow B_b B_b m_a) = 0.19$ .

Similarly, the following counter examples contradict EGL-validity of the schemes in parts 2-6:

$$\begin{aligned} (2) \quad & V_{(0.5,0)}(\neg B_a m_b \rightarrow B_a(\neg B_a m_b)) = 0.64 \\ (3) \quad & V_{(0,0)}(B_b(m_a \vee (m_a \rightarrow m_b)) \rightarrow B_b m_a \vee B_b(m_a \rightarrow m_b)) = 0.19 \\ (4) \quad & V_{(0,0)}(\neg B_b m_a \rightarrow B_b \neg m_a) = 0.19 \\ (5) \quad & V_{(0.5,0)}(B_b m_a \vee B_b \neg m_a) = 0.19 \\ (6) \quad & V_{(1,0)}(m_a \rightarrow B_b \neg B_b \neg m_a) = 0.28 \end{aligned} \quad \square$$

*Remark 2.6.* Note that *positive introspection* and *negative introspection* are not EGL-valid, by Proposition 2.5.

**Proposition 2.7.** *Let  $\varphi$  be an EGL-formula. The following formulas are valid in all EGL-models.*

$$\begin{aligned} (1) \quad & B_a \varphi \wedge B_a(\varphi \rightarrow \psi) \rightarrow B_a \psi \\ (2) \quad & B_a(\varphi \rightarrow \psi) \rightarrow (B_a \varphi \rightarrow B_a \psi) \\ (3) \quad & \neg \neg B_a \neg \neg \varphi \rightarrow \neg \neg B_a \varphi \\ (4) \quad & B_a(\varphi \wedge \psi) \leftrightarrow B_a \varphi \wedge B_a \psi. \end{aligned}$$

*Proof.* We only show that the formula in part (1) is EGL-valid. The validity of the other parts can be shown similarly. Suppose that  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  is an EGL-model and  $s \in S$ . We show that  $B_a \varphi \wedge B_a(\varphi \rightarrow \psi) \rightarrow B_a \psi$  is valid in the pointed model  $(M, s)$ . Let  $\Gamma = \{s' \in S \mid V_{s'}(\varphi) > V_{s'}(\psi)\}$ , then

$$V_s(B_a(\varphi \rightarrow \psi)) = \begin{cases} \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\psi)\} & \Gamma \neq \phi \\ 1 & \Gamma = \phi \end{cases}$$

If  $\Gamma = \phi$ , then  $V_s(B_a\varphi \wedge B_a(\varphi \rightarrow \psi)) = V_s(B_a\varphi)$  and also  $\forall t \in S \quad V_t(\varphi) \leq V_t(\psi)$ . So, it can be obtained that  $\forall s \in S \quad V_s(B_a\varphi) \leq V_s(B_a\psi)$ . Hence,  $V_s(B_a\varphi \wedge B_a(\varphi \rightarrow \psi)) \leq V_s(B_a\psi)$  and so  $V_s(B_a\varphi \wedge B_a(\varphi \rightarrow \psi) \rightarrow B_a\psi) = 1$ . Otherwise, if  $\Gamma \neq \phi$  then  $V_s(B_a\varphi \wedge B_a(\varphi \rightarrow \psi))$

$$\begin{aligned} &= \min \left\{ \min \left\{ \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\}, \min_{s' \in \Gamma^c} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\} \right\}, \right. \\ &\quad \left. \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\psi)\} \right\} \\ &= \min \left\{ \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\}, \min_{s' \in \Gamma^c} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\}, \right. \\ &\quad \left. \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\psi)\} \right\}. \end{aligned}$$

Now, If  $s' \in \Gamma$ , then  $V_{s'}(\psi) < V_{s'}(\varphi)$  and so

$$\min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\psi)\} \leq \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\}$$

Therefore,

$$\begin{aligned} V_s(B_a\varphi \wedge B_a(\varphi \rightarrow \psi)) &= \\ &\min \left\{ \min_{s' \in \Gamma} \max\{1 - r_a(s, s'), V_{s'}(\psi)\}, \min_{s' \in \Gamma^c} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\} \right\}. \end{aligned}$$

Also, if  $s' \in \Gamma^c$ , then  $V_{s'}(\varphi) \leq V_{s'}(\psi)$  and so

$$\min_{s' \in \Gamma^c} \max\{1 - r_a(s, s'), V_{s'}(\varphi)\} \leq \min_{s' \in \Gamma^c} \max\{1 - r_a(s, s'), V_{s'}(\psi)\}$$

Consequently, it can be concluded that  $V_s(B_a\varphi \wedge B_a(\varphi \rightarrow \psi)) \leq V_s(B_a\psi)$ , which completes the proof.  $\square$

*Remark 2.8.* The scheme  $B_a(\varphi) \rightarrow \varphi$  is valid in all reflexive EGL-models.

**2.2. The proof systems.** Let  $\varphi, \psi$  be EGL-formulas and  $a \in \mathcal{A}$ . Consider the following axiom schemes and inference rules:

(A1) all instantiations of propositional Gödel logic tautologies,

(A2)  $B_a\varphi \wedge B_a(\varphi \rightarrow \psi) \rightarrow B_a\psi$

(A3)  $\neg\neg B_a\neg\neg\varphi \rightarrow \neg\neg B_a\varphi$

(A4)  $B_a\varphi \rightarrow \varphi$

(R1)  $\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{ (MP)}$

(R2)  $\frac{\varphi}{B_a\varphi} \text{ (B)}$

We consider a proof system  $B_F$  which has (A1), (A2) and (A3) as its

axiom schemes and both inference rules ( $R1$ ) and ( $R2$ ). The system  $T_F$  is an extension of  $B_F$  by the extra axiom scheme (A4). The axiom (A4), intuitively means that *the completely known facts are completely true*.

Note that the formulas in the first axiom (A1) are not only in the language of propositional Gödel logic, but in the language of EGL.

From now on we assume that  $D$  is the system  $B_F$  or  $T_F$ , unless otherwise stated. A *derivation* of a formula  $\varphi$  from a set of formulas  $\Gamma$  within system  $D$  is defined naturally; notation  $\Gamma \vdash_D \varphi$ . If the system  $D$  is clear from the context, we just write  $\Gamma \vdash \varphi$ .

**Definition 2.9.** Let  $\varphi, \varphi_1, \dots, \varphi_n$  be formulas in the language of EGL.

- (1)  $\varphi$  is  $D$ -consistent if  $\not\vdash_D \neg\varphi$ ,
- (2) A finite set  $\{\varphi_1, \dots, \varphi_n\}$  is  $D$ -consistent if  $\varphi_1 \wedge \dots \wedge \varphi_n$  is  $D$ -consistent,
- (3) An infinite set  $\Gamma$  of formulas is  $D$ -consistent if any finite subset of  $\Gamma$  is  $D$ -consistent,
- (4) A formula or a set of formulas is called  $D$ -inconsistent if it is not  $D$ -consistent,
- (5) A set  $\Gamma$  of formulas is maximally  $D$ -consistent if:
  - (a)  $\Gamma$  is  $D$ -consistent,
  - (b)  $\Gamma \cup \{\psi\}$  is  $D$ -inconsistent for any formula  $\psi \notin \Gamma$ .

If there is no ambiguity, we say *consistent/inconsistent* instead of  $D$ -consistent/ $D$ -inconsistent.

**Lemma 2.10.** (1) Every consistent set of formulas can be extended to a maximally consistent set. (2) Let  $\Gamma$  be a maximally consistent set of formulas, then the following statements hold for all EGL-formulas  $\varphi, \psi$ :

- (a) either  $\varphi \in \Gamma$  or  $\neg\varphi \in \Gamma$ ,
- (b)  $\varphi \wedge \psi \in \Gamma \Leftrightarrow \varphi \in \Gamma$  and  $\psi \in \Gamma$ ,
- (c) if  $\varphi \in \Gamma$  and  $\varphi \rightarrow \psi \in \Gamma$  then  $\psi \in \Gamma$ ,
- (d)  $\Gamma$  is closed under deduction, i.e. if  $\Gamma \vdash \varphi$  then  $\varphi \in \Gamma$ .

*Proof.* The proof is similar to the proof of Lemma 1.4.3 of [19].  $\square$

### 3. SOUNDNESS AND COMPLETENESS

Let  $\mathcal{M}$  be a class of EGL-models. The system  $D$  is called:

- *sound* with respect to  $\mathcal{M}$ , if  $\vdash \varphi$  then  $\mathcal{M} \models \varphi$ ,
- *complete* with respect to  $\mathcal{M}$ , if  $\mathcal{M} \models \varphi$  then  $\vdash \varphi$ ,
- *weak complete* with respect to  $\mathcal{M}$ , if  $\mathcal{M} \models \varphi$  then  $\vdash \neg\neg\varphi$ .

**Lemma 3.1.** *The inference rules (R1) and (R2) are admissible, i.e. for each rule if all premises are EGL-valid, then its conclusion is also EGL-valid.*

*Proof.* The proof is obvious.  $\square$

**Theorem 3.2. (Soundness)**

- (1)  $B_F$  is sound with respect to the class of all EGL-models
- (2)  $T_F$  is sound with respect to the class of all reflexive EGL-models.

*Proof.* The proof is obtained in a straightforward manner by applying Proposition 2.7, Remark 2.8 and Lemma 3.1.  $\square$

**Lemma 3.3. (Model Existence Lemma)** *The following statements are equivalent:*

- (1) *If  $\models \varphi$ , then  $\vdash \neg\neg\varphi$ ,*
- (2) *if  $\not\models \neg\neg\varphi$  then there is an EGL-model  $M = (S, r_{a|_{a \in A}}, \pi)$  and a state  $s \in S$  such that  $(M, s) \not\models \varphi$ ,*
- (3) *if  $\neg\neg\varphi$  is consistent then it is satisfiable, i.e. there is an EGL-model  $M = (S, r_{a|_{a \in A}}, \pi)$  and a state  $s \in S$  such that  $(M, s) \models \neg\neg\varphi$ .*

*Proof.* (1) and (2) are obviously equivalent. For convention, we use the notation " $\exists M \exists s$ " to mean "there exists an EGL-model  $M = (S, r_{a|_{a \in A}}, \pi)$  and a state  $s \in S$ ".

(2) $\Rightarrow$ (3). We restate the statements (2) and (3) as follows, respectively:

$$\begin{aligned} \not\models \neg\neg\varphi &\implies \exists M \exists s V_s(\varphi) \neq 1 \\ \not\models \neg\neg\neg\varphi &\implies \exists M \exists s V_s(\neg\neg\varphi) = 1 \end{aligned}$$

Assume that (2) holds, then

$$\not\models \neg\neg\varphi \implies \exists M \exists s V_s(\neg\varphi) = \begin{cases} 0 & 0 < V_s(\varphi) < 1 \\ 1 & V_s(\varphi) = 0 \end{cases}$$

Replacing  $\varphi$  by  $\neg\varphi$  gives rise to

$$\not\models \neg\neg\neg\varphi \implies \exists M \exists s V_s(\neg\neg\varphi) = \begin{cases} 0 & 0 < V_s(\neg\varphi) < 1 \\ 1 & V_s(\neg\varphi) = 0 \end{cases}$$

But since  $\neg\varphi$  takes crisp value, the case where  $0 < V_s(\neg\varphi) < 1$  never happens. Therefore,

$$\not\models \neg\neg\neg\varphi \implies \exists M \exists s V_s(\neg\neg\varphi) = 1$$

(3) $\Rightarrow$ (2). Assume that (3) holds, then  $\not\models \neg\neg\neg\varphi$  implies that  $\exists M \exists s V_s(\neg\varphi) = 0$ . Replacing  $\varphi$  by  $\neg\varphi$ , it can be obtained that

$$\not\models \neg\neg\neg\neg\varphi \implies \exists M \exists s V_s(\neg\neg\varphi) = 0$$

Therefore,  $\not\models \neg\neg\varphi$  implies that  $\exists M \exists s V_s(\varphi) \neq 1$ , which completes the proof.  $\square$



**Theorem 3.4. (Weak Completeness)**  $B_F$  is weak complete with respect to the class of all EGL-models.

*Proof.* By model existence lemma, it is enough to show that for each formula  $\varphi$ , if  $\neg\neg\varphi$  is consistent then  $\neg\neg\varphi$  is satisfiable. By Lemma 2.10 part (1), it is sufficient to show that for each maximally consistent set  $\Phi$  of formulas, the set of all double negated formulas contained in  $\Phi$  is satisfiable. This is obtained by constructing a so-called canonical EGL-model. The canonical EGL-model  $M^c = (S^c, r_{a \in \mathcal{A}}^c, \pi^c)$  is defined as follows:

- $S^c = \{s_\Theta \mid \Theta \text{ is a maximally consistent set of formulas}\}$
- $r_{a \in \mathcal{A}}^c(s_\Theta, s_\Psi) = \begin{cases} 1 & \Theta/B_a \subseteq \Psi \\ 0 & \text{otherwise} \end{cases}$ , where  $\Theta/B_a = \{\neg\neg\varphi \mid \neg\neg B_a\varphi \in \Theta\}$ ,
- $\pi^c(s_\Theta, p) = \begin{cases} 1 & \neg\neg p \in \Theta \\ 0 & \neg\neg p \notin \Theta \end{cases}$ , where  $p \in \mathcal{P}$ .

$\pi^c$  can be extended to all formulas, naturally. As before we use the notation  $V$  for its extension. Note that  $\pi^c$  and  $V$  take crisp values. Now it is enough to show that for each formula  $\varphi$  and each  $s_\Theta \in S^c$  the following statement holds:

$$V_{s_\Theta}(\neg\neg\varphi) = 1 \Leftrightarrow \neg\neg\varphi \in \Theta$$

We prove it by induction on the structure of  $\varphi$ :

**Case 1:**  $\varphi = p$ , where  $p \in \mathcal{P}$ :

$$V_{s_\Theta}(\neg\neg p) = 1 \Leftrightarrow V_{s_\Theta}(p) > 0 \Leftrightarrow V_{s_\Theta}(p) = 1 \Leftrightarrow \pi^c(s_\Theta, p) = 1 \Leftrightarrow \neg\neg p \in \Theta$$

**Case 2:**  $\varphi = \varphi_1 \wedge \varphi_2$

$$\begin{aligned} V_{s_\Theta}(\neg\neg(\varphi_1 \wedge \varphi_2)) = 1 &\Leftrightarrow V_{s_\Theta}(\varphi_1 \wedge \varphi_2) > 0 \Leftrightarrow V_{s_\Theta}(\varphi_1) > 0, V_{s_\Theta}(\varphi_2) > 0 \Leftrightarrow \\ V_{s_\Theta}(\neg\neg\varphi_1) = 1, V_{s_\Theta}(\neg\neg\varphi_2) = 1 &\Leftrightarrow \neg\neg\varphi_1, \neg\neg\varphi_2 \in \Theta \quad (\text{induction hypothesis}) \\ \Leftrightarrow \neg\neg\varphi_1 \wedge \neg\neg\varphi_2 \in \Theta &\quad (\text{maximal consistency of } \Theta) \\ \Leftrightarrow \neg\neg(\varphi_1 \wedge \varphi_2) \in \Theta &\quad ((\text{GT4}) \text{ and maximal consistency of } \Theta) \end{aligned}$$

**Case 3:**  $\varphi = \varphi_1 \rightarrow \varphi_2$

( $\Leftarrow$ ) Let  $\neg\neg(\varphi_1 \rightarrow \varphi_2) \in \Theta$ , then using (GT5) and maximal consistency of  $\Theta$  it is concluded that

$$\neg\neg\varphi_1 \rightarrow \neg\neg\varphi_2 \in \Theta \tag{3.1}$$

Assume that  $V_{s_\Theta}(\neg\neg(\varphi_1 \rightarrow \varphi_2)) = 0$ . Then,  $V_{s_\Theta}(\varphi_1 \rightarrow \varphi_2) = 0$  and so it is demonstrated that  $V_{s_\Theta}(\varphi_1) = 1$ ,  $V_{s_\Theta}(\varphi_2) = 0$ . Hence  $V_{s_\Theta}(\neg\neg\varphi_1) = 1$  and  $V_{s_\Theta}(\neg\neg\varphi_2) = 0$ . By induction hypothesis,  $\neg\neg\varphi_1 \in \Theta$  and then by

applying Lemma 2.10, the statement (3.1) and the maximal consistency of  $\Theta$ , it is concluded that  $\neg\neg\varphi_2 \in \Theta$ . Consequently, by induction hypothesis  $V_{s_\Theta}(\neg\neg\varphi_2) = 1$ , contradicting the assumption.

( $\Rightarrow$ ) Assume that  $V_{s_\Theta}(\neg\neg(\varphi_1 \rightarrow \varphi_2)) = 1$ , then  $V_{s_\Theta}(\varphi_1 \rightarrow \varphi_2) = 1$ . Thus it is obtained that either  $V_{s_\Theta}(\varphi_2) = 1$  or  $V_{s_\Theta}(\varphi_1) = V_{s_\Theta}(\varphi_2) = 0$ . If  $V_{s_\Theta}(\varphi_2) = 1$ , it is derived that  $V_{s_\Theta}(\neg\neg\varphi_2) = 1$  and then by induction hypothesis  $\neg\neg\varphi_2 \in \Theta$ . Since  $\neg\neg\varphi_2 \rightarrow (\neg\neg\varphi_1 \rightarrow \neg\neg\varphi_2)$  is an instance of (GT1), then maximal consistency of  $\Theta$  results in  $\neg\neg\varphi_1 \rightarrow \neg\neg\varphi_2 \in \Theta$ . Therefore, by (GT5) we obtain  $\neg\neg(\varphi_1 \rightarrow \varphi_2) \in \Theta$ . In the other case, if  $V_{s_\Theta}(\varphi_1) = V_{s_\Theta}(\varphi_2) = 0$ , by absurd hypothesis suppose that  $\neg\neg(\varphi_1 \rightarrow \varphi_2) \notin \Theta$ . Hence, by maximal consistency of  $\Theta$  it is obtained that  $\neg(\varphi_1 \rightarrow \varphi_2) \in \Theta$ . Then by applying (GT8) and (GT6),  $\neg\varphi_1 \rightarrow \varphi_2 \in \Theta$  and  $\neg\neg(\neg\varphi_1 \rightarrow \varphi_2) \in \Theta$ , respectively. Now, by case 3, we have  $V_{s_\Theta}(\neg\neg(\neg\varphi_1 \rightarrow \varphi_2)) = 1$  and hence  $V_{s_\Theta}(\neg\varphi_1 \rightarrow \varphi_2) = 1$ , contradicting the assumption  $V_{s_\Theta}(\varphi_1) = V_{s_\Theta}(\varphi_2) = 0$ .

**Case 4:**  $\varphi = B_a\psi$

( $\Leftarrow$ ) Assume that  $\neg\neg B_a\psi \in \Theta$ , then  $\neg\neg\psi \in \Theta/B_a$ . Let  $\Psi$  be an arbitrary maximally consistent set of formulas and  $s_\Psi \in S^c$ . Then we have:

$$\begin{aligned} r_a^c(s_\Theta, s_\Psi) = 1 &\Rightarrow \neg\neg\psi \in \Theta/B_a \subseteq \Psi \Rightarrow V_{s_\Psi}(\neg\neg\psi) = 1 \quad (\text{induction hypothesis}) \\ &\Rightarrow V_{s_\Psi}(\psi) = 1 \end{aligned}$$

Consequently,  $V_{s_\Theta}(B_a\psi) = \min_{s_\Psi \in S^c} \max \{1 - r_a^c(s_\Theta, s_\Psi), V_{s_\Psi}(\psi)\} = 1$ , so  $V_{s_\Theta}(\neg\neg B_a\psi) = 1$ .

( $\Rightarrow$ ) Assume that  $V_{s_\Theta}(\neg\neg B_a\psi) = 1$ .

CLAIM:  $\Theta/B_a \cup \{\neg\neg\neg\psi\}$  is inconsistent.

*Proof of the Claim.* Suppose that  $\Theta/B_a \cup \{\neg\neg\neg\psi\}$  is consistent. Then by lemma 2.10 there exists a maximally consistent extension  $\Psi$  such that  $\Theta/B_a \subseteq \Theta/B_a \cup \{\neg\neg\neg\psi\} \subseteq \Psi$ . Thus,  $r_a^c(s_\Theta, s_\Psi) = 1$  and  $\neg\neg\neg\psi \in \Psi$ . By induction hypothesis  $V_{s_\Psi}(\neg\neg\neg\psi) = 1$  and then  $V_{s_\Psi}(\psi) = 0$ . Therefore,

$$V_{s_\Theta}(B_a\psi) = \min_{s_\Psi \in S^c} \max \{1 - r_a^c(s_\Theta, s_\Psi), V_{s_\Psi}(\psi)\} = 0$$

and it is concluded that  $V_{s_\Theta}(\neg\neg B_a\psi) = 0$ , which contradicts the assumption.  $\blacksquare$  *Claim*

Consequently, it follows that there is some inconsistent finite subset  $\Delta$  of  $\Theta/B_a \cup \{\neg\neg\neg\psi\}$ , let  $\Delta = \{\neg\neg\varphi_1, \dots, \neg\neg\varphi_k, \neg\neg\neg\psi\}$ . Note that without loss of generality, we assume that  $\Delta$  contains  $\neg\neg\neg\psi$ . Hence

$$\vdash \neg(\neg\neg\varphi_1 \wedge \dots \wedge \neg\neg\varphi_k \wedge \neg\neg\neg\psi).$$

By (GT4), (GT6) and transitivity rule it can be obtained that

$$\vdash (\varphi_1 \wedge \dots \wedge \varphi_k \wedge \neg\psi) \rightarrow (\neg\neg\varphi_1 \wedge \dots \wedge \neg\neg\varphi_k \wedge \neg\neg\neg\psi)$$

So applying (GT2) yields

$$\vdash \neg(\neg\neg\varphi_1 \wedge \dots \wedge \neg\neg\varphi_k \wedge \neg\neg\neg\psi) \rightarrow \neg(\varphi_1 \wedge \dots \wedge \varphi_k \wedge \neg\psi)$$

Hence, it can be concluded that  $\vdash \neg(\varphi_1 \wedge \dots \wedge \varphi_k \wedge \neg\psi)$ , and then  $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_k \rightarrow \neg\neg\psi) \dots))$ , by using (GT3). Consequently by applying (R2) we obtain  $\vdash B_a(\varphi_1 \rightarrow \xi)$ , where  $\xi$  is  $\varphi_2 \rightarrow (\dots (\varphi_k \rightarrow \neg\neg\psi) \dots)$ . Then  $\vdash \neg\neg B_a(\varphi_1 \rightarrow \xi)$ , by (GT6). Since  $\Theta$  is a maximally consistent set, then by Lemma 2.10 the following statement holds

$$\neg\neg B_a(\varphi_1 \rightarrow \xi) \in \Theta \quad (3.2)$$

Also from  $\neg\neg\varphi_1, \dots, \neg\neg\varphi_k \in \Theta/B_a$ , it is demonstrated that

$$\neg\neg B_a\varphi_1, \dots, \neg\neg B_a\varphi_k \in \Theta \quad (3.3)$$

Thus, we have

$$\begin{aligned} & \vdash B_a\varphi_1 \wedge B_a(\varphi_1 \rightarrow \xi) \rightarrow B_a\xi && \text{(instance of (A2))} \\ & \vdash B_a\varphi_1 \rightarrow (B_a(\varphi_1 \rightarrow \xi) \rightarrow B_a\xi) && \text{(by (G4))} \\ & \vdash \neg\neg(B_a\varphi_1 \rightarrow (B_a(\varphi_1 \rightarrow \xi) \rightarrow B_a\xi)) && \text{(by (GT6))} \\ & \vdash \neg\neg B_a\varphi_1 \rightarrow (\neg\neg B_a(\varphi_1 \rightarrow \xi) \rightarrow \neg\neg B_a\xi) && \text{(by (GT5))} \end{aligned}$$

Therefore,

$$\neg\neg B_a\varphi_1 \rightarrow (\neg\neg B_a(\varphi_1 \rightarrow \xi) \rightarrow \neg\neg B_a\xi) \in \Theta. \quad (3.4)$$

By (3.3), (3.4) and the maximal consistency of  $\Theta$  it follows that:

$$\neg\neg B_a(\varphi_1 \rightarrow \xi) \rightarrow \neg\neg B_a\xi \in \Theta \quad (3.5)$$

Also, by (3.2), (3.5) and the maximal consistency of  $\Theta$  it can be concluded that  $\neg\neg B_a\xi \in \Theta$ . Repetitive application of the above process yields  $\neg\neg B_a\neg\neg\psi \in \Theta$  and then by using (A3) we conclude that  $\neg\neg B_a\psi \in \Theta$ , which completes the proof.  $\square$

**Theorem 3.5.**  *$T_F$  is weak complete with respect to the class of all reflexive EGL-models.*

*Proof.* The proof is similar to the proof of Theorem 3.4, except for the indistinguishing functions of canonical model. We must consider the following condition in our canonical model:

$$\forall a \in \mathcal{A} \ \forall s \in S^c \ r_a^c(s, s) = 1 \quad \square$$

#### 4. EGL WITH COMMON KNOWLEDGE

Let  $\mathcal{B} \subseteq \mathcal{A}$ . We add two connectives  $E_{\mathcal{B}}\varphi$  and  $C_{\mathcal{B}}\varphi$  to the language of EGL, which mean "every agents in  $\mathcal{B}$  knows  $\varphi$ " and " $\varphi$  is common knowledge between the agents of  $\mathcal{B}$ ", respectively. We call this extension the *epistemic Gödel logic with common knowledge*, denoted by C EGL.

**Definition 4.1.** The C EGL-formulas are defined as follows:

$$\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid B_a \varphi \mid E_{\mathcal{B}} \varphi \mid C_{\mathcal{B}} \varphi$$

where,  $p \in \mathcal{P}$  and  $a \in \mathcal{A}$ . Notice that the connective  $E_{\mathcal{B}}$  can be defined in terms of  $B_a$ s, that is  $E_{\mathcal{B}}(\varphi) = \bigwedge_{a \in \mathcal{B}} B_a \varphi$ .

**Definition 4.2. (CEGL-Model)** A CEGL-model is a structure  $M = (S, r_{a|a \in \mathcal{A}}, \pi)$ , which its parameters are defined as an EGL-model, but the extension  $V$  of  $\pi$  is also defined for two new connectives  $E_{\mathcal{B}}$  and  $C_{\mathcal{B}}$  as follows:

- $V_s(E_{\mathcal{B}} \varphi) = \min_{b \in \mathcal{B}} V_s(B_b \varphi)$
- $V_s(C_{\mathcal{B}} \varphi) = \inf_{i \in \mathbb{N}} E_{\mathcal{B}}^i \varphi$

where, for each  $i \in \mathbb{N}$ ,  $E_{\mathcal{B}}^i \varphi = E_{\mathcal{B}} E_{\mathcal{B}} \dots E_{\mathcal{B}} \varphi$  (i times).

**Proposition 4.3.** *The following rules are admissible in CEGL, that is if the premise of a rule is CEGL-valid, then its conclusion is also CEGL-valid.*

- (1)  $\frac{\varphi}{E_{\mathcal{B}} \varphi} (E)$
- (2)  $\frac{\varphi}{C_{\mathcal{B}} \varphi} (C)$

*Proof.* The proof is straightforward.  $\square$

**Lemma 4.4.** *Let  $\varphi, \psi$  be formulas in the language of CEGL. The following statements hold:*

- (1)  $E_{\mathcal{B}} \varphi \rightarrow \varphi$  is valid in all reflexive CEGL-models.
- (2)  $C_{\mathcal{B}} \varphi \rightarrow \varphi$  is valid in all reflexive CEGL-models.
- (3) For each  $n \in \mathbb{N}$ , the formula  $E_{\mathcal{B}}^n(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^n \varphi \rightarrow E_{\mathcal{B}}^n \psi$  is CEGL-valid.
- (4)  $E_{\mathcal{B}}^n(\varphi \rightarrow \psi) \rightarrow (E_{\mathcal{B}}^n \varphi \rightarrow E_{\mathcal{B}}^n \psi)$  is CEGL-valid.

*Proof.* Parts (1) and (2) can be proved easily. Parts (3) and (4) are equivalent, by using (G4). So we only prove the part (3). Let  $M = (S, r_{a|a \in \mathcal{A}}, \pi)$  be a CEGL-model and  $s \in S$ . We show by induction on  $n$  that  $V_s(E_{\mathcal{B}}^n(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^n \varphi \rightarrow E_{\mathcal{B}}^n \psi) = 1$ .

(Basis) By (A2),  $V_s(B_b(\varphi \rightarrow \psi) \wedge B_b \varphi \rightarrow B_b \psi) = 1$ , for each  $b \in \mathcal{B}$ . Therefore,

$$\begin{aligned} & \min \left\{ V_s(B_b(\varphi \rightarrow \psi)), V_s(B_b \varphi) \right\} \leq V_s(B_b \psi) \\ \implies & \min_{b \in \mathcal{B}} \left\{ \min \left\{ V_s(B_b(\varphi \rightarrow \psi)), V_s(B_b \varphi) \right\} \right\} \leq \min_{b \in \mathcal{B}} \left\{ V_s(B_b \psi) \right\} \\ \implies & \min \left\{ \min_{b \in \mathcal{B}} V_s(B_b(\varphi \rightarrow \psi)), \min_{b \in \mathcal{B}} V_s(B_b \varphi) \right\} \leq \min_{b \in \mathcal{B}} V_s(B_b \psi) \\ \implies & V_s(E_{\mathcal{B}}(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}} \varphi) \leq V_s(E_{\mathcal{B}} \psi) \end{aligned}$$

and so  $V_s(E_{\mathcal{B}}(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}\varphi \rightarrow E_{\mathcal{B}}\psi) = 1$ .

(*Induction step*) Assume that  $V_s(E_{\mathcal{B}}^i(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^i\varphi \rightarrow E_{\mathcal{B}}^i\psi) = 1$ , for each  $1 \leq i \leq n$ . We check that  $V_s(E_{\mathcal{B}}^{n+1}(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^{n+1}\varphi \rightarrow E_{\mathcal{B}}^{n+1}\psi) = 1$ . By induction hypothesis and the admissibility of (R2) the following equation is obtained:

$$V_s(B_{\mathcal{B}}(E_{\mathcal{B}}^n(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^n\varphi \rightarrow E_{\mathcal{B}}^n\psi)) = 1$$

Using Proposition 2.7 part (ii), it is demonstrated that for each  $b \in \mathcal{B}$ :

$$V_s(B_b(E_{\mathcal{B}}^n(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^n\varphi \rightarrow E_{\mathcal{B}}^n\psi)) = 1$$

Also, by Proposition (2.7), part (iv), we obtain the following equation for each  $b \in \mathcal{B}$ :

$$V_s(B_b E_{\mathcal{B}}^n(\varphi \rightarrow \psi) \wedge B_b E_{\mathcal{B}}^n\varphi \rightarrow B_b E_{\mathcal{B}}^n\psi) = 1$$

Therefore, for all  $b \in \mathcal{B}$ :

$$\begin{aligned} \min \{V_s(B_b E_{\mathcal{B}}^n(\varphi \rightarrow \psi)), V_s(B_b E_{\mathcal{B}}^n\varphi)\} &\leq V_s(B_b E_{\mathcal{B}}^n\psi), \\ \min_{b \in \mathcal{B}} \{ \min \{V_s(B_b E_{\mathcal{B}}^n(\varphi \rightarrow \psi)), V_s(B_b E_{\mathcal{B}}^n\varphi)\} \} &\leq \min_{b \in \mathcal{B}} V_s(B_b E_{\mathcal{B}}^n\psi), \\ \min \{ \min_{b \in \mathcal{B}} V_s(B_b E_{\mathcal{B}}^n(\varphi \rightarrow \psi)), \min_{b \in \mathcal{B}} V_s(B_b E_{\mathcal{B}}^n\varphi) \} &\leq \min_{b \in \mathcal{B}} V_s(B_b E_{\mathcal{B}}^n\psi), \\ \min \{V_s(E_{\mathcal{B}}^{n+1}(\varphi \rightarrow \psi)), V_s(E_{\mathcal{B}}^{n+1}\varphi)\} &\leq \min V_s(E_{\mathcal{B}}^{n+1}\psi), \end{aligned}$$

and consequently

$$V_s(E_{\mathcal{B}}^{n+1}(\varphi \rightarrow \psi) \wedge E_{\mathcal{B}}^{n+1}\varphi \rightarrow E_{\mathcal{B}}^{n+1}\psi) = 1. \quad \square$$

**Corollary 4.5.** *For each  $n \in \mathbb{N}$ ,  $E_{\mathcal{B}}^n(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}^n\varphi$  is valid in all reflexive CEGL-models.*

*Proof.* Let  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  be a reflexive CEGL-model and  $s \in S$ . We show by induction on  $n$  that  $(E_{\mathcal{B}}^n(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}^n\varphi)$  is valid in  $(M, s)$ .

(*Basis*) By Lemma 4.4 part (i), the formula  $E_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow (\varphi \rightarrow E_{\mathcal{B}}\varphi)$  is valid in all reflexive CEGL-models, so  $V_s(E_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \wedge \varphi \rightarrow E_{\mathcal{B}}\varphi) = 1$ . Since by Lemma 4.4 part (i),  $V_s(E_{\mathcal{B}}\varphi) \leq V_s(\varphi)$ , then we have  $V_s(E_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi)) \leq V_s(E_{\mathcal{B}}\varphi)$ . Therefore,  $V_s(E_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}\varphi) = 1$ .

(*Induction step*) Assume that  $V_s(E_{\mathcal{B}}^n(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}^n\varphi) = 1$ , then we show that  $V_s(E_{\mathcal{B}}^{n+1}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}^{n+1}\varphi) = 1$ . By induction hypothesis and Proposition 4.3, it is obtained that  $V_s(E_{\mathcal{B}}(E_{\mathcal{B}}^n(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}^n\varphi)) = 1$ . So  $V_s(E_{\mathcal{B}}^{n+1}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow E_{\mathcal{B}}^{n+1}\varphi) = 1$ , by using Lemma 4.4 part (iv).  $\square$

**Proposition 4.6.** *The following formulas are valid in all reflexive CEGL-models.*

- (1)  $E_{\mathcal{B}}\varphi \rightarrow B_a\varphi$
- (2)  $C_{\mathcal{B}}\varphi \rightarrow E_{\mathcal{B}}\varphi$
- (3)  $C_{\mathcal{B}}(\varphi \rightarrow \psi) \wedge C_{\mathcal{B}}\varphi \rightarrow C_{\mathcal{B}}\psi$
- (4)  $C_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow (\varphi \rightarrow C_{\mathcal{B}}\varphi)$

*Proof.* The parts (1) and (2) can be shown easily. Let  $M = (S, r_{a|a \in \mathcal{A}}, \pi)$  be a reflexive CEGL-model and  $s \in S$ . We show (3) and (4) are valid in  $(M, s)$ .

*Proof of (3).* Using Lemma 4.4 we have

$$\begin{aligned} \min \left\{ V_s(C_{\mathcal{B}}\varphi \rightarrow \psi), V_s(C_{\mathcal{B}}\varphi) \right\} &= \min \left\{ \inf_{i \in \mathbb{N}} V_s(E_{\mathcal{B}}^i\varphi \rightarrow \psi), \inf_{i \in \mathbb{N}} V_s(E_{\mathcal{B}}^i\varphi) \right\} \\ &\leq \inf_{i \in \mathbb{N}} \min \left\{ V_s(E_{\mathcal{B}}^i\varphi \rightarrow \psi), V_s(E_{\mathcal{B}}^i\varphi) \right\} \leq \inf_{i \in \mathbb{N}} V_s(E_{\mathcal{B}}^i\psi) = V_s(C_{\mathcal{B}}\psi) \end{aligned}$$

Hence,  $V_s(C_{\mathcal{B}}(\varphi \rightarrow \psi) \wedge C_{\mathcal{B}}\varphi \rightarrow C_{\mathcal{B}}\psi) = 1$

*Proof of (4).* By Corollary 4.5,  $V_s(E_{\mathcal{B}}^i(\varphi \rightarrow E_{\mathcal{B}}\varphi)) \leq V_s(E_{\mathcal{B}}^i\varphi)$ , for all  $i \in \mathbb{N}$ . Then,  $\inf_{i \in \mathbb{N}} V_s(E_{\mathcal{B}}^i(\varphi \rightarrow E_{\mathcal{B}}\varphi)) \leq \inf_{i \in \mathbb{N}} V_s(E_{\mathcal{B}}^i\varphi)$ , so

$$V_s(C_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi)) \leq V_s(C_{\mathcal{B}}\varphi).$$

Therefore,  $V_s(C_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \wedge \varphi) \leq V_s(C_{\mathcal{B}}\varphi)$  and consequently

$$V_s(C_{\mathcal{B}}(\varphi \rightarrow E_{\mathcal{B}}\varphi) \rightarrow (\varphi \rightarrow C_{\mathcal{B}}\varphi)) = 1 \quad \square$$

*Remark 4.7.* The formula  $C_{\mathcal{B}}(\varphi \rightarrow \psi) \wedge C_{\mathcal{B}}\varphi \rightarrow C_{\mathcal{B}}\psi$ , in part (3) of Proposition 4.6, is also valid in all non-reflexive CEGL-models.

## 5. A DYNAMIC EXTENSION OF EGL

**Definition 5.1. (Action Model)** Let  $\mathcal{L}$  be any logical language. An action model  $A$  is a structure  $A = (E, u_{a|a \in \mathcal{A}}, pre)$  such that

- $E$  is a non-empty finite set of actions,
- for each  $a \in \mathcal{A}$ ,  $u_a : E \times E \rightarrow [0, 1]$  is a (indistinguishing) function, which for each  $e \in E$ ,  $u_a(e, e) = 1$ ,
- $pre : E \rightarrow \mathcal{L}$  is a function which assigns a precondition  $pre(e) \in \mathcal{L}$ , to each action  $e \in E$ . We denote  $pre(e)$  by  $pre_e$ .

**Definition 5.2.** The set of formulas of *dynamic epistemic Gödel logic* (briefly, DEGL) are defined as follows:

$$\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid B_a\varphi \mid [A, e]\varphi$$

where,  $A = (E, u_{a|a \in \mathcal{A}}, pre)$  is an action model whose language  $\mathcal{L}$  is defined by the following BNF:

$$\chi ::= G(\psi) = g \mid G(\psi) > g \mid \perp \mid \chi \wedge \chi \mid \chi \rightarrow \chi$$

in which,

$$\psi ::= p \mid \perp \mid \psi \wedge \psi \mid \psi \rightarrow \psi \mid B_a\psi$$

where,  $p \in \mathcal{P}$ ,  $a \in \mathcal{A}$ ,  $e \in E$  and  $g \in [0, 1]$ . We call  $[A, e]\varphi$  the *updating* formula and we refer to  $G(\psi) = g$  and  $G(\psi) > g$  as  $G$ -formulas.

Intuitively,  $G(\psi) = g$  says that the truth value of  $\psi$  is  $g$  and  $G(\psi) > g$  says that the truth value of  $\psi$  is strictly greater than  $g$ .

In the language  $\mathcal{L}$ , the formulas  $\neg\varphi$ ,  $\varphi \vee \psi$  and  $\varphi \leftrightarrow \psi$  are defined as similar as GL. Moreover, further connectives are defined as follows:

$$\begin{aligned} G(\varphi) \geq g &:= G(\varphi) = g \vee G(\varphi) > g \\ G(\varphi) \leq g &:= \neg(G(\varphi) > g) \\ G(\varphi) < g &:= \neg(G(\varphi) \geq g) \end{aligned}$$

**Definition 5.3.** A DEGL-model is a structure  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  in which  $S$ ,  $r_{a|_{a \in \mathcal{A}}}$  and  $\pi$  are defined analogous to their definition in the EGL-models, but the expansion  $V$  of  $\pi$  is also defined on updating and  $G$ -formulas as follows:

$$\begin{aligned} \bullet V_s(G(\varphi) = g) &= \begin{cases} 1 & V_s(\varphi) = g \\ 0 & \text{otherwise} \end{cases} \\ \bullet V_s(G(\varphi) > g) &= \begin{cases} 1 & V_s(\varphi) > g \\ 0 & \text{otherwise} \end{cases} \\ \bullet V_s([A, e]\varphi) &= \begin{cases} V'_{(s,e)}(\varphi) & (s, e) \in S' \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

where,  $V'_{(s,e)}(\varphi)$  is defined below. Also note that the  $G$ -formulas take crisp values.

**Definition 5.4. (Product Model)** Let  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  be a DEGL-model and  $A = (E, u_{a|_{a \in \mathcal{A}}}, pre)$  be an action model. Then, *updating  $M$  by  $A$*  is defined by the product model  $M \times A = (S', r'_{a|_{a \in \mathcal{A}}}, \pi')$ , where:

- $S' = \{(s, e) \mid s \in S, e \in E, V_s(pre_e) = 1\}$  is the set of states,
- for each  $a \in \mathcal{A}$ ,  $r'_a : S' \times S' \rightarrow [0, 1]$  is a function, which for all  $(s_1, e_1), (s_2, e_2) \in S'$ :  

$$r'_a((s_1, e_1), (s_2, e_2)) = \min \{r_a(s_1, s_2), u_a(e_1, e_2)\}$$

We call it the indistinguishing function,

- $\pi' : S' \times \mathcal{P} \rightarrow [0, 1]$  is a valuation function, which for each  $(s, e) \in S'$  and each  $p \in \mathcal{P}$ , we have  $\pi'((s, e), p) = \pi(s, p)$ .

The function  $\pi'$  can be extended to all DEGL-formulas, naturally. We use the similar notation  $V'$  for the extended function.

**Proposition 5.5.** *Let  $\varphi$  be a DEGL-formula. The following formulas are valid in all DEGL-models.*

- (1)  $[A, e]p \leftrightarrow (pre_e \rightarrow p)$
- (2)  $[A, e]\neg\varphi \leftrightarrow (pre_e \rightarrow \neg[A, e]\varphi)$
- (3)  $[A, e](\varphi \wedge \psi) \leftrightarrow ([A, e]\varphi \wedge [A, e]\psi)$
- (4)  $[A, e]B_a\varphi \rightarrow (pre_e \rightarrow B_a[A, e]\varphi)$
- (5)  $\bigwedge_{e' \in E, u_a(e, e') \neq 0} B_a[A, e']\varphi \rightarrow [A, e]B_a\varphi$

*Proof.* We only prove parts (4) and (5). Other parts can be proved in a similar fashion more easily. Assume that  $M = (S, r_{a|_{a \in \mathcal{A}}}, \pi)$  is a DEGL-model,  $A = (E, u_{a|_{a \in \mathcal{A}}}, pre)$  is an action model,  $M \times A = (S', r'_{a|_{a \in \mathcal{A}}}, \pi')$  is the product model corresponding to the update of  $M$  by  $A$ ,  $s \in S$  and  $e \in E$ .

*Proof of (4)* If  $V_s(pre_e) = 1$ , then

$$\begin{aligned}
 V_s([A, e]B_a\varphi) &= V'_{(s,e)}(B_a\varphi) \\
 &= \min_{(s',e') \in S'} \max\{1 - r'_a((s, e), (s', e')), V'_{(s',e')}(\varphi)\} \\
 &= \min_{(s',e') \in S'} \max\{1 - \min\{r_a(s, s'), u_a(e, e')\}, V'_{(s',e')}(\varphi)\} \\
 &= \min_{(s',e') \in S'} \max\{\max\{1 - r_a(s, s'), 1 - u_a(e, e')\}, V'_{(s',e')}(\varphi)\} \\
 &= \min_{(s',e') \in S'} \max\{1 - r_a(s, s'), V'_{(s',e')}(\varphi), 1 - u_a(e, e')\}
 \end{aligned}$$

and so we have

$$\begin{aligned}
 V_s([A, e]B_a\varphi) &= \min \left\{ \min_{s' \in S} \max\{1 - r_a(s, s'), V'_{(s',e)}(\varphi), 1 - u_a(e, e)\}, \right. \\
 &\quad \left. \min_{(s',e') \in S', e' \neq e} \max\{1 - r_a(s, s'), V'_{(s',e')}(\varphi), 1 - u_a(e, e')\} \right\} \\
 &= \min \left\{ V_s(B_a[A, e]\varphi), \min_{(s',e') \in S', e' \neq e} \max\{1 - r_a(s, s'), V'_{(s',e')}(\varphi), 1 - u_a(e, e')\} \right\}
 \end{aligned}$$

Therefore,  $V_s([A, e]B_a\varphi) \leq V_s(B_a[A, e]\varphi)$  and hence

$$V_s([A, e]B_a\varphi \rightarrow B_a[A, e]\varphi) = 1.$$

Otherwise, if  $V_s(pre_e) = 0$ , then  $V_s([A, e]B_a\varphi \wedge pre_e) \leq V_s(B_a[A, e]\varphi)$ , which completes the proof.

*Proof of (5)* We consider two cases. In first case  $V_s(pre_e) = 1$ , then for each  $e' \in E$  which  $u_a(e, e') \neq 0$ , we have the following equation:

$$\begin{aligned}
 V_s(B_a[A, e']\varphi) &= \min_{s' \in S} \max\{1 - r_a(s, s'), V_{s'}([A, e']\varphi)\} \\
 &= \min_{s' \in S} \max\{1 - r_a(s, s'), V'_{(s',e')}(\varphi)\}
 \end{aligned}$$

Similar to the proof of part (4), it can be obtained that

$$\begin{aligned}
 V_s([A, e]B_a\varphi) &= \min \left\{ \max\{1 - r_a(s, s'), V'_{(s',e')}(\varphi), 1 - u_a(e, e')\} \right. \\
 &\quad \left. \mid s' \in S, e' \in E, u_a(e, e') \neq 0, V_{s'}(pre_{e'}) = 1 \right\}
 \end{aligned}$$



and consequently

$$\begin{aligned}
V_s(\bigwedge_{e' \in E, u_a(e, e') \neq 0} B_a[A, e']\varphi) &= \min_{e' \in E, u_a(e, e') \neq 0} \min_{s' \in S} \max\{1 - r_a(s, s'), V'_{(s', e')}(\varphi)\} \\
&\leq \min_{e' \in E, u_a(e, e') \neq 0} \min_{s' \in S} \max\{1 - r_a(s, s'), V'_{(s', e')}(\varphi), 1 - u_a(e, e')\} \\
&\leq \min_{(s', e') \in S', u_a(e, e') \neq 0} \max\{1 - r_a(s, s'), V'_{(s', e')}(\varphi), 1 - u_a(e, e')\} \\
&= V_s([A, e]B_a\varphi)
\end{aligned}$$

So,  $V_s(\bigwedge_{e' \in E, u_a(e, e') \neq 0} B_a[A, e']\varphi \rightarrow [A, e]B_a\varphi) = 1$ . In the other case  $V_s(pre_e) = 0$ , then by definition it can be obtained that  $V_s([A, e]B_a\varphi) = 1$ , which completes the proof.  $\square$

**Example 5.6. (Updating the fuzzy muddy children)** In this example an update of Example 2.4 of fuzzy muddy children is given. For each agent  $a \in \mathcal{A}$ , we consider a new parameter  $H_a \in [0, 1]$  as the amount of *hearing impairment* of the agent  $a$ . If  $E = \{e_1, \dots, e_m\}$  is the set of actions, then for each  $a \in \mathcal{A}$  and  $1 \leq i, j \leq m$ , the indistinguishing function  $u_a$  is defined as follows:

$$u_a(e_i, e_j) = \begin{cases} H_a & i \neq j \\ 1 & i = j \end{cases}$$

We suppose that  $\mathcal{A} = \{a, b\}$  and somebody makes an announcement about two things: (1) number of agents with muddy face (2) the amount of mud in each muddy face. Then,  $E = \{e_1, e_2\}$  while  $pre_{e_1}$  says "The face of exactly one of the agents is at least a little muddy" and  $pre_{e_2}$  says "The face of exactly one of the agents is a little muddy". We consider the value  $\frac{1}{2}$  as being a little muddy. So, the pre-conditions are the following formulas:

$$\begin{aligned}
pre_{e_1} &:= (G(m_a) \geq \frac{1}{2} \vee G(m_b) \geq \frac{1}{2}) \wedge \neg(G(m_a) \geq \frac{1}{2} \wedge G(m_b) \geq \frac{1}{2}) \\
pre_{e_2} &:= (G(m_a) = \frac{1}{2} \vee G(m_b) = \frac{1}{2}) \wedge \neg(G(m_a) = \frac{1}{2} \wedge G(m_b) = \frac{1}{2})
\end{aligned}$$

Also, we suppose that  $H_a = 0, H_b = 0.75$ . Then the indistinguishing functions  $u_a, u_b$  are defined as follows:

$$u_a(e_i, e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad u_b(e_i, e_j) = \begin{cases} 0.75 & i \neq j \\ 1 & i = j \end{cases}$$

Thus, the definition of action model  $A = (E, u_{a|a \in \mathcal{A}}, pre)$  is completely done. Now we update the model  $M$  in Example 2.4 by the action model  $A$ . The product model  $M \times A = (S', r'_{a|a \in \mathcal{A}}, \pi')$  is defined as follows:

- The set of states  $S'$  is as follows:

$$S' = \{((0, \frac{1}{2}), e_1), ((0, 1), e_1), ((\frac{1}{2}, 0), e_1), ((1, 0), e_1), \\ , ((0, \frac{1}{2}), e_2), ((\frac{1}{2}, 0), e_2), ((1, \frac{1}{2}), e_2), ((\frac{1}{2}, 1), e_2)\}$$

- Let  $(i, j), (i', j') \in S'$  and  $k \in \{1, 2\}$ . Then the indistinguishing functions are defined as below:

$$r'_a(((i, j), e_k), ((i', j'), e_k)) = \begin{cases} 0 & |j - j'| = 0 \\ 0.36 & |j - j'| = \frac{1}{2} \\ 0.32 & |j - j'| = 1 \end{cases}$$

$$r'_a(((i, j), e_1), ((i', j'), e_2)) = 0$$

$$r'_b(((i, j), e_k), ((i', j'), e_k)) = \begin{cases} 0 & |i - i'| = 0 \\ 0.81 & |i - i'| = \frac{1}{2} \\ 0.72 & |i - i'| = 1 \end{cases}$$

$$r'_b(((i, j), e_1), ((i', j'), e_2)) = \begin{cases} 0.75 & |i - i'| = 0 \\ 0.75 & |i - i'| = \frac{1}{2} \\ 0.72 & |i - i'| = 1 \end{cases}$$

- The valuation function on atomic propositions  $m_a, m_b$  are defined as follows:

$$\pi'((i, j), e_k), m_a = i \\ \pi'((i, j), e_k), m_b = j$$

where,  $(i, j) \in S'$  and  $k \in \{1, 2\}$ .

## CONCLUSIONS

We introduced some epistemic extensions of Gödel logic. We established two deductive systems which are sound and weak complete with respect to the corresponding Kripke semantics. Afterwards, we enriched the language of epistemic Gödel logic with two operators for group and common Knowledge and proposed the corresponding semantics. Moreover, by introducing an action model approach, we gave a dynamic extension of the epistemic Gödel logic. Also, we derived the validity of some formulas in the language of dynamic extension.

For the future work, if one can extend Proposition 5.5, such that it gives a translation from DEGL to (static) epistemic Gödel logic, then one can prove soundness and weak completeness for DEGL.

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